Convergence limits in perturbation theory

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Introduction

- Perturbation Theory (PT) is a standard method in reactor physics
- Fundamental contributions to PT were given, among others, by Augusto Gandini, who developed what is now called **Generalized Perturbation Theory** (GPT)
- GPT is commonly used for Sensitivity Analysis and Uncertainty Quantification, usually as a **first-order** approach
- Recent advancements in higher-order harmonics computation allow to increase the perturbation order of GPT

Framework and motivations

Why do we need higher-order PT?

Description of non-linear phenomena → e.g. control rods self-shielding, temperature feedback and depletion effects.

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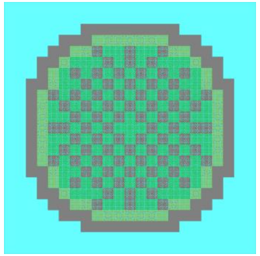
Improving PT accuracy is useful for:

- **reactors in operation**

Gen-III/III+ reactors (EPR)



Framework and motivations

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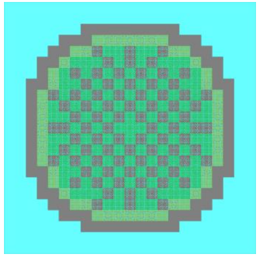
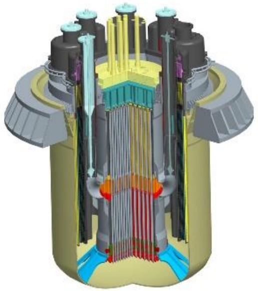
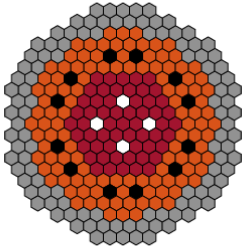
- **reactors in operation**
- **reactors under design**

Gen-III/III+ reactors (EPR)

Gen-IV reactors (ALFRED)

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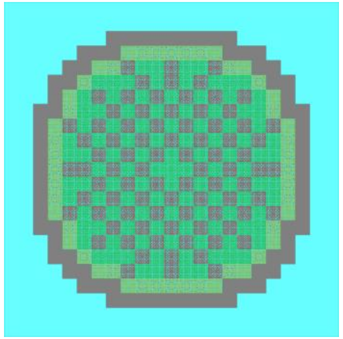

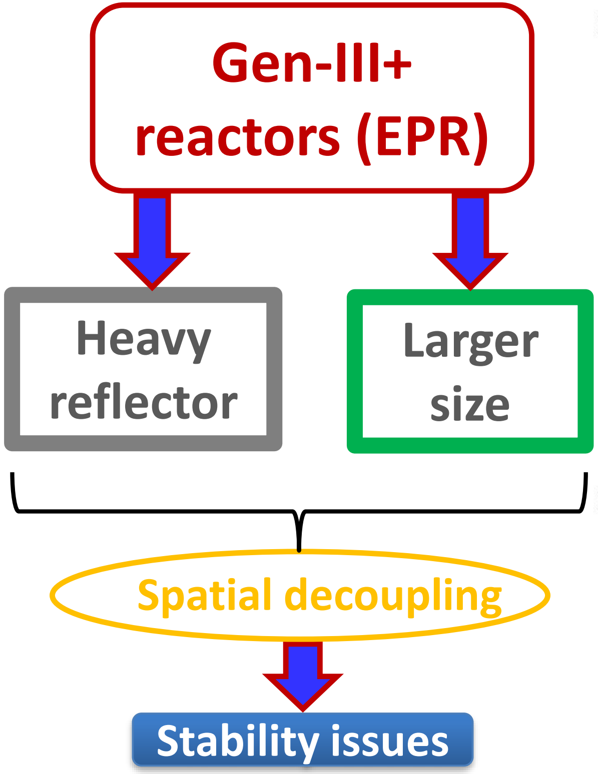

- Large thermal reactors achieve **higher efficiency** by means of leakage reduction, accomplished employing stainless-steel reflector and larger core size;
- These aspects increase the spatial decoupling degree→ *what occurs locally has a negligible influence on the global reactor behaviour!*

How does GPT work?

When the operators of the reference criticality problem (transport or diffusion, continuous or discrete…) are perturbed, they are expressed as a superposition of the reference operators plus some deviations (i.e. the perturbation)

$$\hat{L}\vec{\phi} = \lambda\hat{F}\vec{\phi}$$

Reference eigenproblem

GPT Method

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$\hat{L}' = \hat{L} + \delta\hat{L}$ perturbation

Reference operator

$\hat{F}' = \hat{F} + \delta\hat{F}$ perturbation

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Reference eigenproblem

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perturbation

Reference operator

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perturbation

Reference operator

$$\hat{L}'\vec{\phi}' = \lambda'\hat{F}'\vec{\phi}'$$

Perturbed eigenvalue and eigenstates

GPT Method

- Perturbed criticality problem:

$$\hat{L}'\vec{\phi}' = \lambda'\hat{F}'\vec{\phi}'$$

$\hat{L}'$ and $\hat{F}'$ → direct perturbations

λ' and $\vec{\phi}'$ → indirect perturbations

- GPT assumes indirect perturbations as sums of infinite terms, with $\{\mu_0, \vec{\varphi}_0\}$ the reference system eigenpair :

$$\vec{\phi}' = \vec{\varphi}_0 + \sum_{n=0}^{\infty} \vec{\phi}^{(n)}$$

$$\lambda' = \mu_0 + \sum_{n=0}^{\infty} \lambda^{(n)}$$

Flux and eigenvalue perturbations

GPT Method

- According to the Standard Method formulation (Gandini), each flux perturbation $\phi^{(n)}$ can be expressed as an expansion on the eigenvectors of the reference problem,

$$\vec{\phi}^{(n)} = \sum_{i=0}^{\infty} a_i^{(n)} \vec{\varphi}_i$$

- $a_i^{(n)}$ can be computed via projection on the **adjoint problem**, while $\vec{\varphi}_i$ have to be computed solving $\hat{L}\vec{\varphi}_i = \mu_i \hat{F}\vec{\varphi}_i$
- In general, only a few harmonics $\vec{\varphi}_i$ are used, and they need to be evaluated using **numerical methods**

GPT convergence

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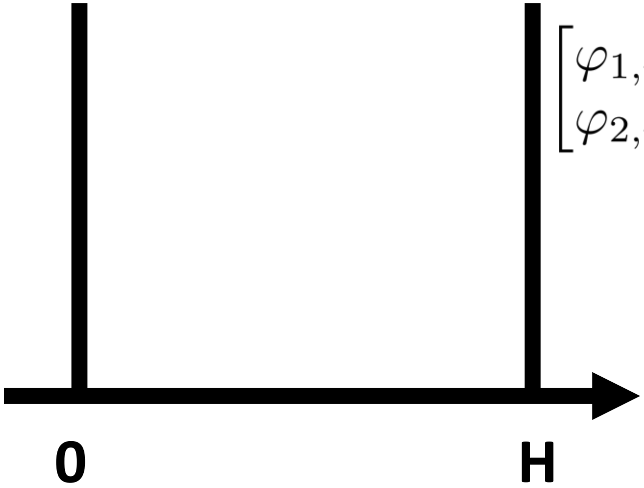
Does GPT always converge?

In other words, can we use GPT whatever the perturbation operator $\delta\hat{A}$ is?

Convergence limits

- To assess, at least qualitatively, GPT convergence limits, we referred to a very simple model problem, two-group diffusion in a purely thermal slab ($\chi_1 = 1, \Sigma_{f,1} = 0$);



$$\begin{bmatrix} \varphi_{1,i} \\ \varphi_{2,i} \end{bmatrix} = \begin{bmatrix} \frac{\Sigma_{r,2}(1+L_2^2 B_i^2)}{\Sigma_{1\to 2}} \sin(\mathrm{B_i x}) \\ \sin(\mathrm{B_i x}) \end{bmatrix}$$

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1. Amplitude, δ
2. **Spatial width, Δx**

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1. Amplitude, δ
2. Spatial width, Δx
3. **Position, $\boldsymbol{x_0}$**

Perturbation choice

- Perturbation of single entries of the leakage and multiplication operators are considered

$$\delta\hat{L} = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{dx}}\delta D_1 \frac{\mathrm{d}}{\mathrm{dx}} * + \delta\Sigma_{r,1} & 0 \\ -\delta\Sigma_{1\to2} & -\frac{\mathrm{d}}{\mathrm{dx}}\delta D_2 \frac{\mathrm{d}}{\mathrm{dx}} * + \delta\Sigma_{r,2} \end{pmatrix}$$

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NEMO

Capture perturbation

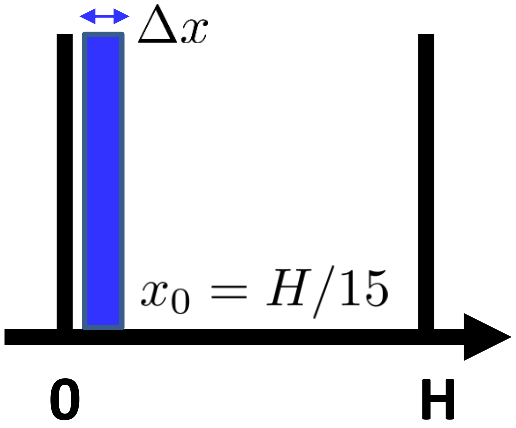
POLITECNICO DI TORINO

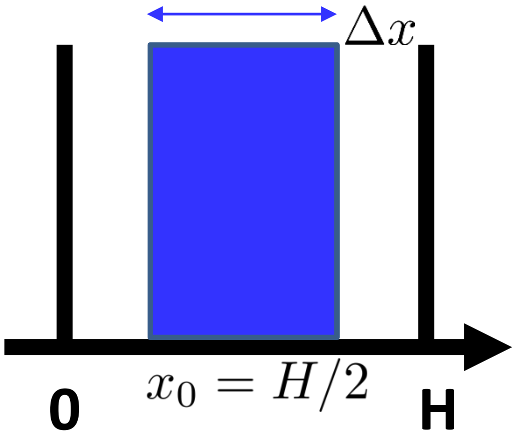
- $\Delta x = H$ (whole core perturbation)→ **analytic case**
- The flux shape does not change
- The eigenvalue changes

$$\lambda' = \frac{\left(1+\frac{D_1}{\Sigma'_{r,1}}B^2\right)\left(1+\frac{D_2}{\Sigma_{r,2}}B^2\right)}{k_\infty} = \mu_0\left(1+\frac{\delta}{1+L_1^2B^2}\right)$$

- The perturbed eigenvalue depends linearly on the perturbation amplitude;
- GPT converges exactly at 1st order (no truncation error)

Capture perturbation





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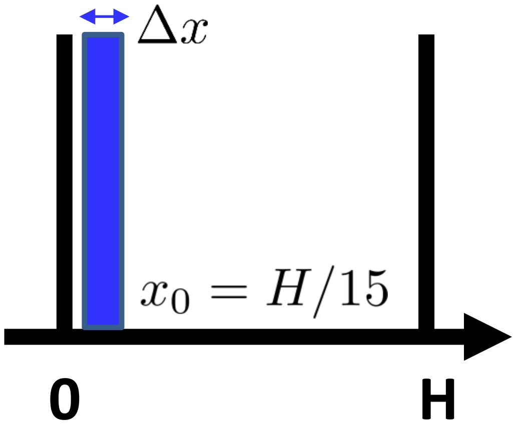
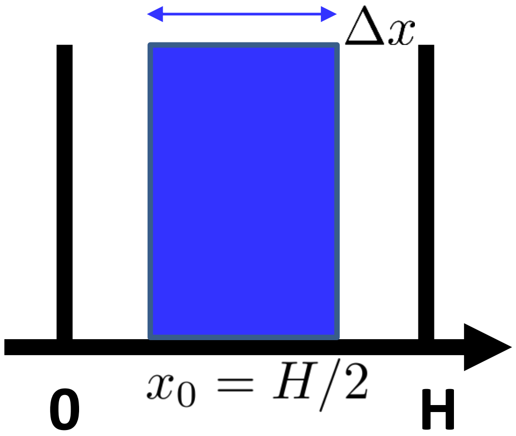




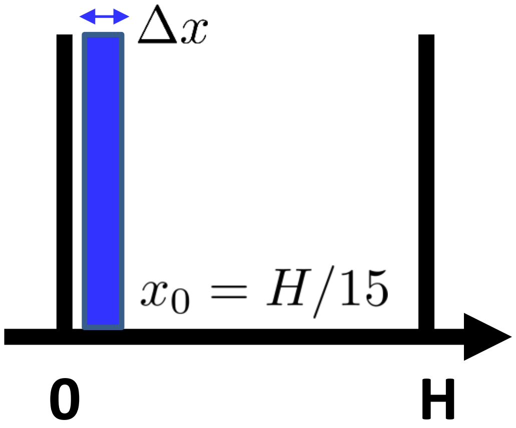
All the local perturbations introduced lead to consistent sets of perturbed multi-group cross sections

$$\Sigma'_{c,g} = \Sigma_{c,g} + \delta\Sigma_{c,g}$$

$$\Sigma'_{a,g} = \Sigma_{a,g} + \delta\Sigma_{c,g}$$

$$\Sigma'_{t,g} = \Sigma_{t,g} + \delta\Sigma_{c,g} \rightarrow D'_g = D_g + \delta D_g$$

Capture perturbation



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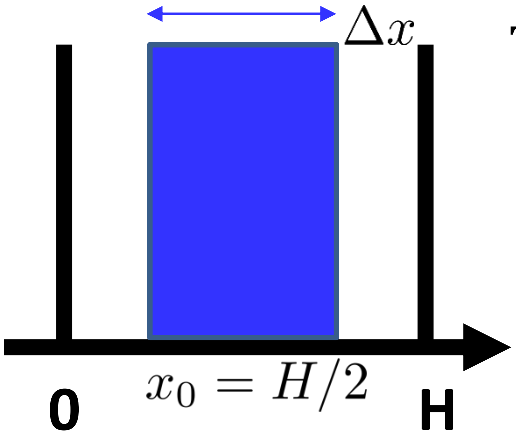
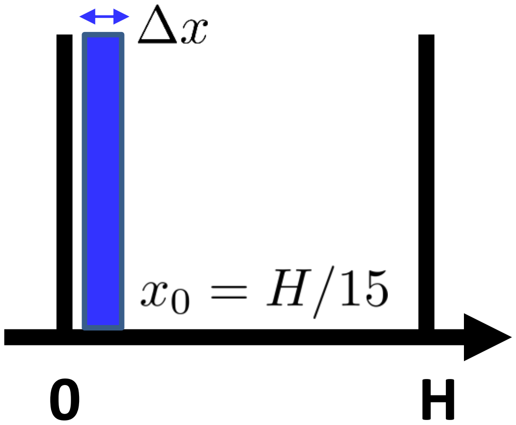


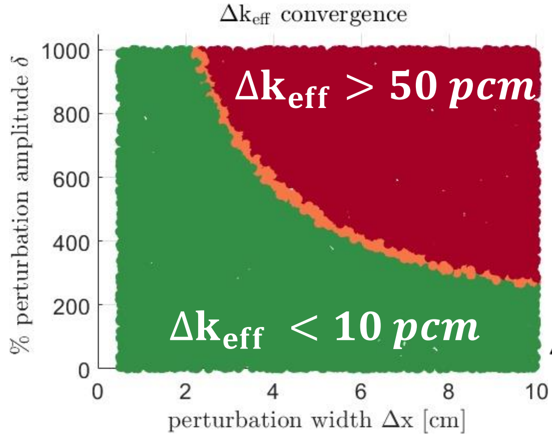
The series are truncated to finite sums

$N = 20$ $M = 100$

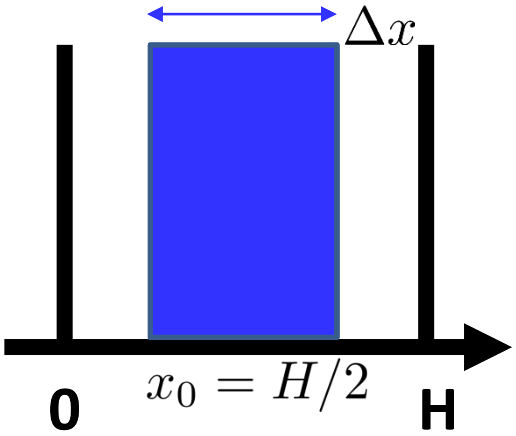
$$\vec{\phi}' = \vec{\varphi}_0 + \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} a_i^{(n)} \vec{\varphi}_i$$

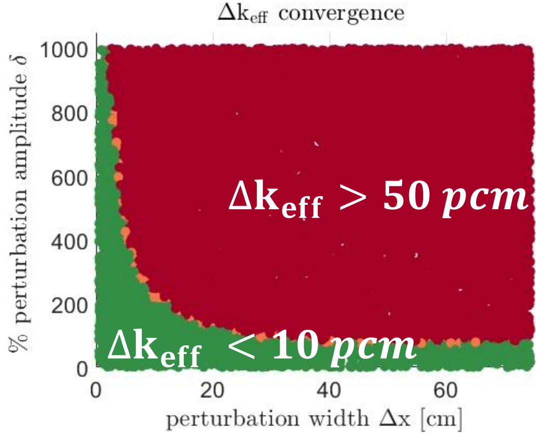
Capture perturbation





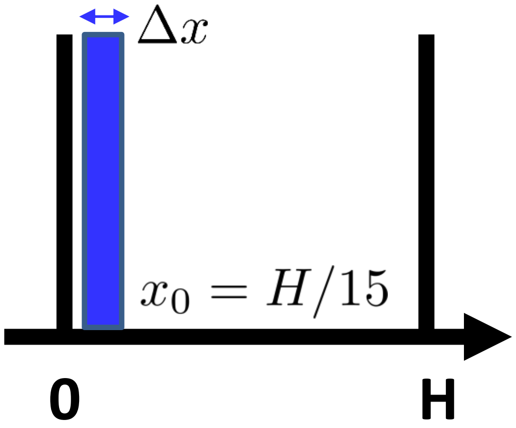
$$\Delta k_{eff} = |k_{eff,GPT} - k_{eff,comp}|$$

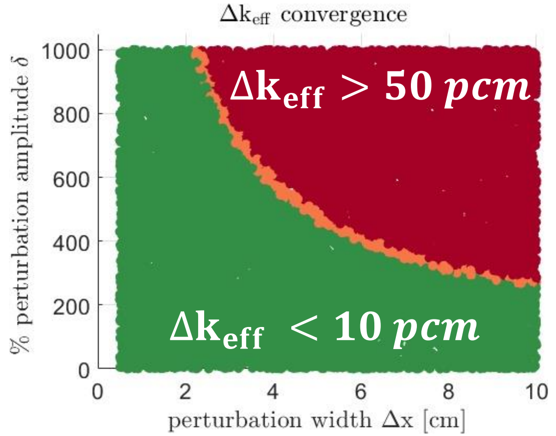




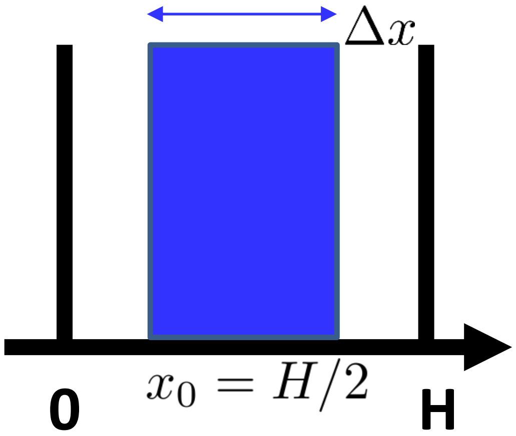
$$\Delta \phi = ||\phi_{GPT} - \phi_{comp}||$$

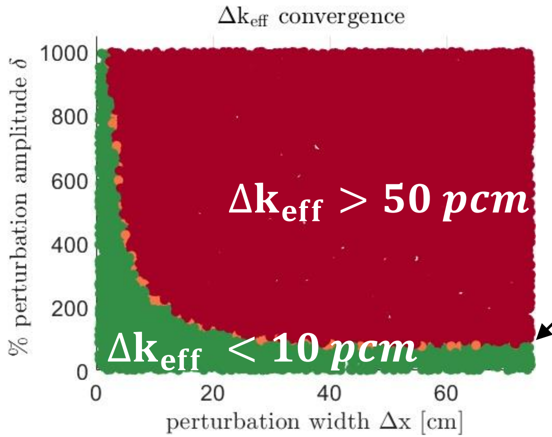
Capture perturbation





$$\delta \Delta x = \mathrm{const}$$





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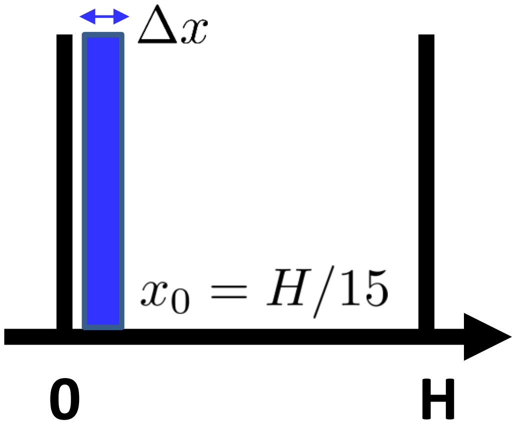

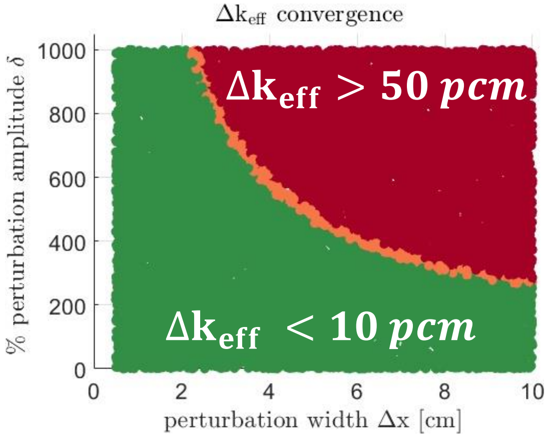

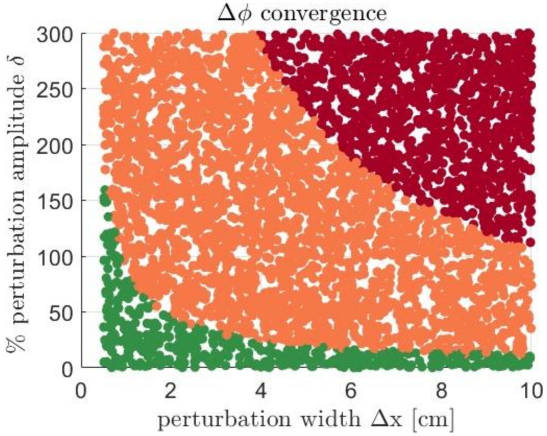

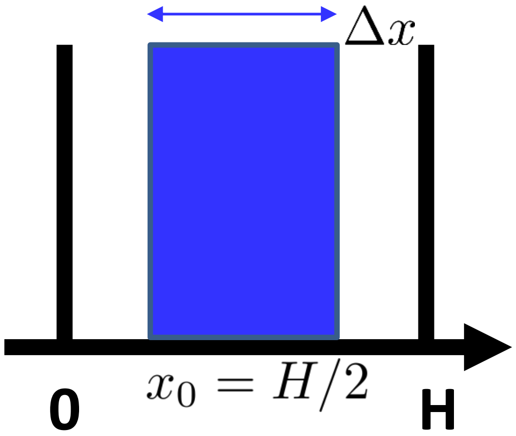

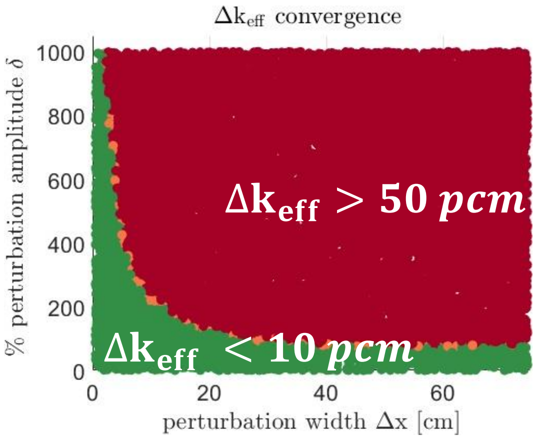

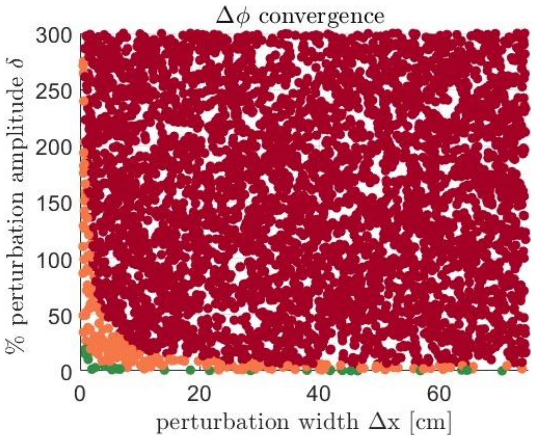

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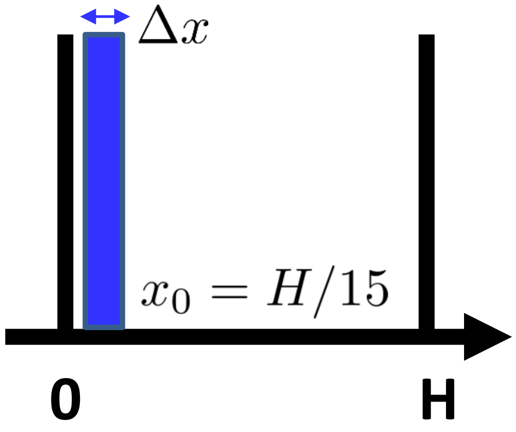

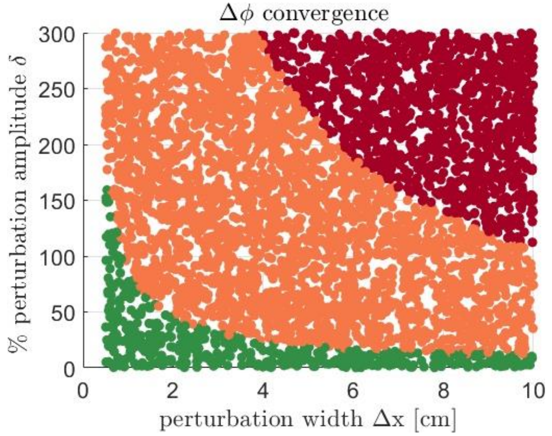

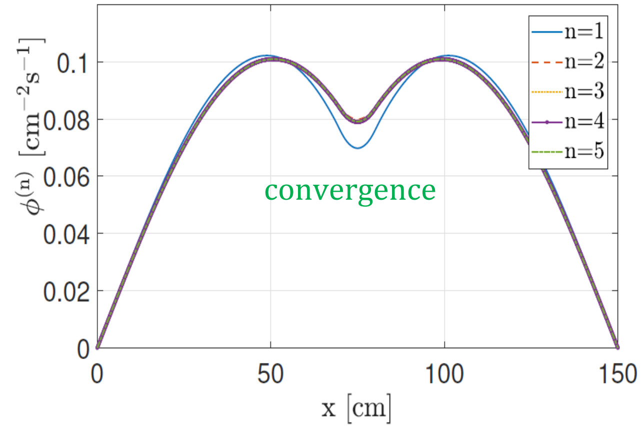

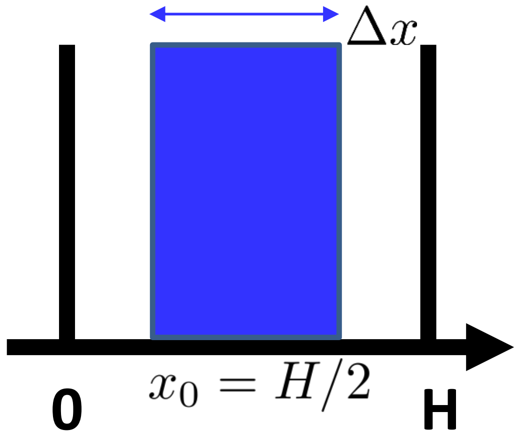

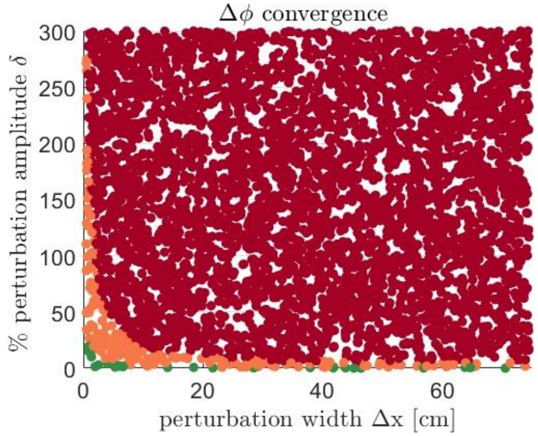

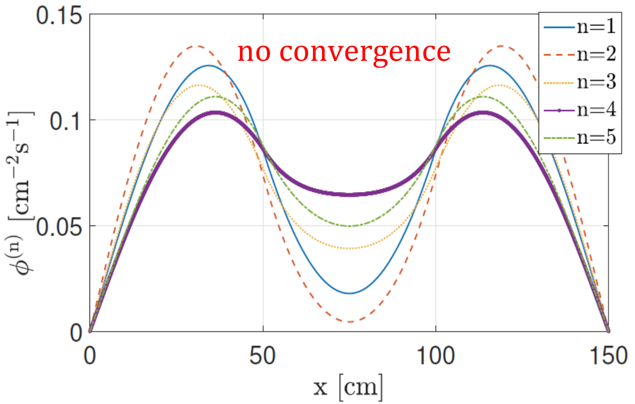

Neutron emission perturbation

$$\delta\hat{L} = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{dx}}\delta D_1 \frac{\mathrm{d}}{\mathrm{dx}} * + \delta\Sigma_{r,1} & 0 \\ -\delta\Sigma_{1\to 2} & -\frac{\mathrm{d}}{\mathrm{dx}}\delta D_2 \frac{\mathrm{d}}{\mathrm{dx}} * + \delta\Sigma_{r,2} \end{pmatrix}$$

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Neutron emission perturbation

- $\Delta x = H$ (whole core perturbation)→ **analytic case**
- The flux shape does not change, but the eigenvalue does!

$$\lambda' = \frac{(1+L_1^2B^2)(1+L_2^2B^2)}{k'_\infty} = \frac{\Sigma_1\Sigma_2(1+L_1^2B^2)(1+L_2^2B^2)}{\nu\Sigma_{f,2}\Sigma_{1\to2}(1+\delta)} = \frac{\mu_0}{1+\delta}$$

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Reference eigenvalue

Perturbation intensity

Taking the Taylor expansion of this expression with respect to δ, we get

$$\lambda' = \mu_0 \sum_{n=0}^{\infty} (-1)^n \delta^n$$

Convergent only for $|\delta| < 1$

Neutron emission perturbation

$$\lambda' = \mu_0 \sum_{n=0}^{\infty} (-1)^n \delta^n$$

IF these two series were the same, GPT would have Taylor series convergence limit, $|\delta| < 1$

$$\lambda' = \lambda^{(0)} + \lambda^{(1)} + ..\lambda^{(\infty)} = \lambda^{(0)} + \sum_{n=1}^{\infty} \lambda^{(n)} = \mu_0 + \sum_{n=1}^{\infty} \lambda^{(n)}$$

Neutron emission perturbation

- Using GPT formulas, the first three perturbation terms are

$$\lambda^{(1)} = -\mu_0\delta,\ \lambda^{(2)} = \mu_0\delta^2,\ \lambda^{(3)} = -\mu_0\delta^3$$

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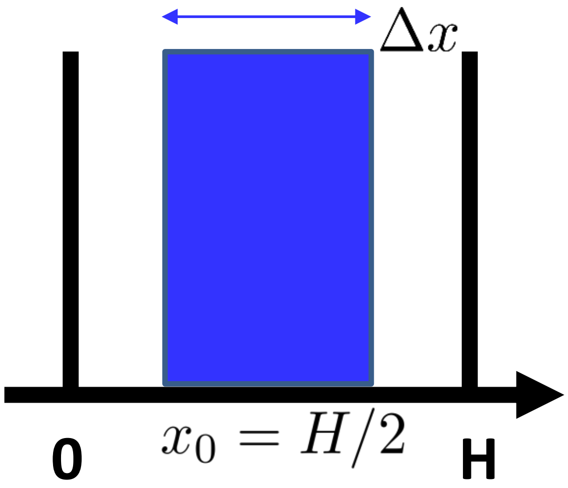
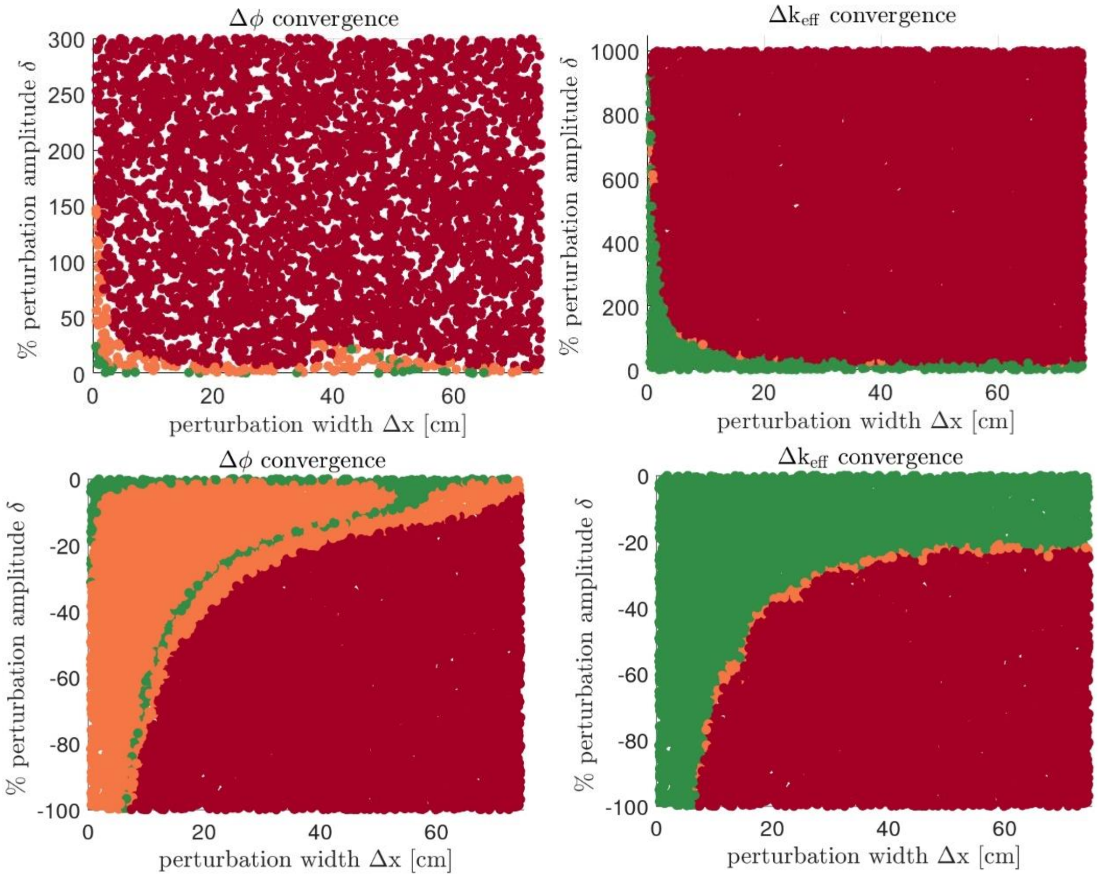
The perturbed eigenvalue **diverges** if the perturbation amplitude modulus is larger or equal 1 (>= 100%)!

Neutron emission perturbation





Thermal fission perturbation

$$\delta \hat{L} = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{dx}}\delta D_1 \frac{\mathrm{d}}{\mathrm{dx}} * + \delta\Sigma_{r,1} & 0 \\ -\delta\Sigma_{1\to 2} & -\frac{\mathrm{d}}{\mathrm{dx}}\delta D_2 \frac{\mathrm{d}}{\mathrm{dx}} * + \delta\Sigma_{r,2} \end{pmatrix}$$

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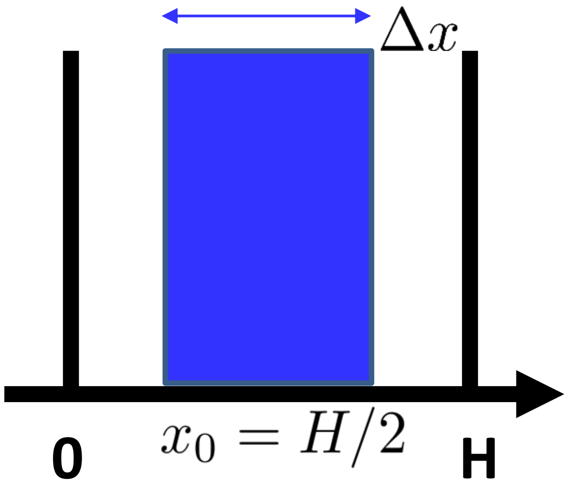

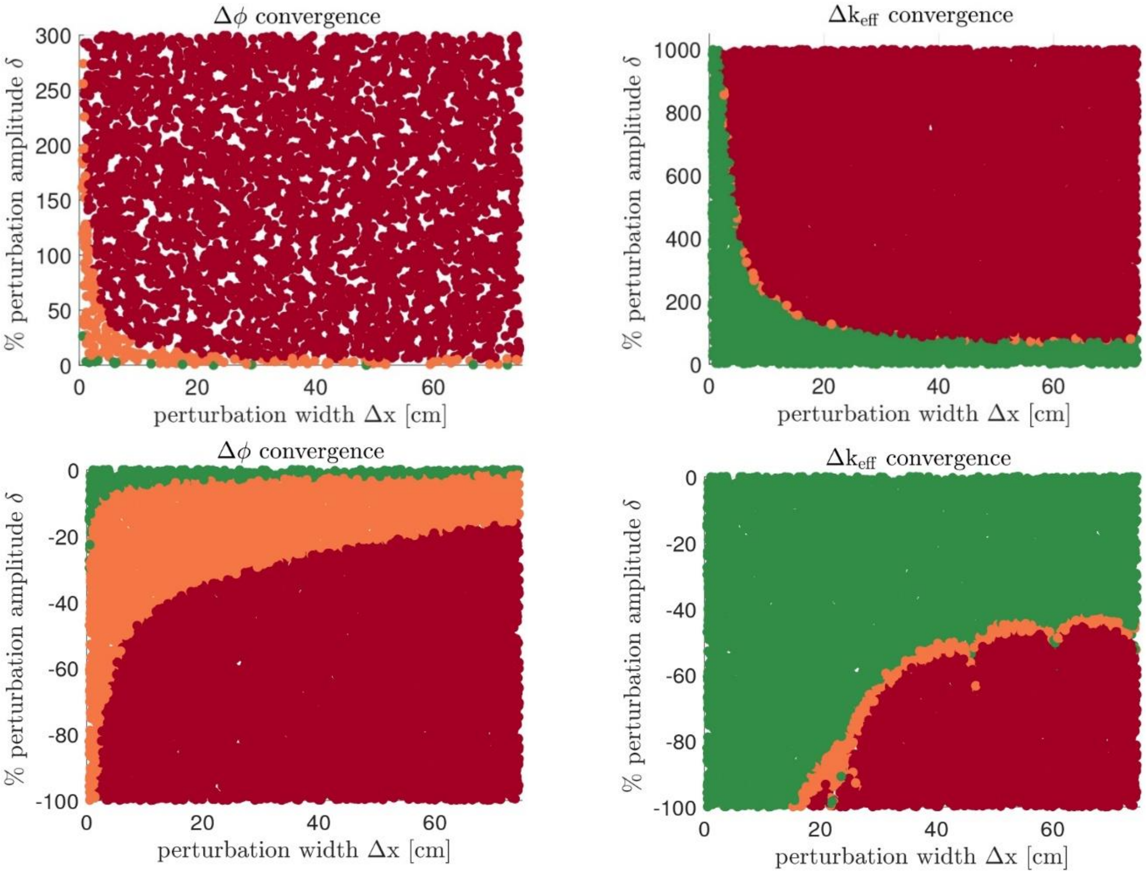

Fast fission perturbation

$$\delta\hat{L} = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{dx}}\delta D_1 \frac{\mathrm{d}}{\mathrm{dx}} * + \delta\Sigma_{r,1} & 0 \\ -\delta\Sigma_{1\to 2} & -\frac{\mathrm{d}}{\mathrm{dx}}\delta D_2 \frac{\mathrm{d}}{\mathrm{dx}} * + \delta\Sigma_{r,2} \end{pmatrix}$$

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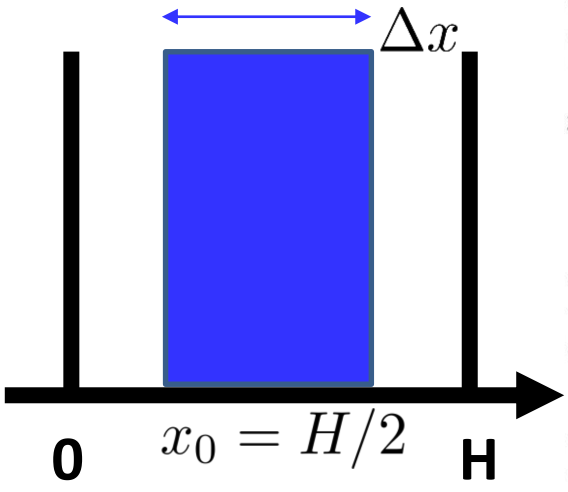
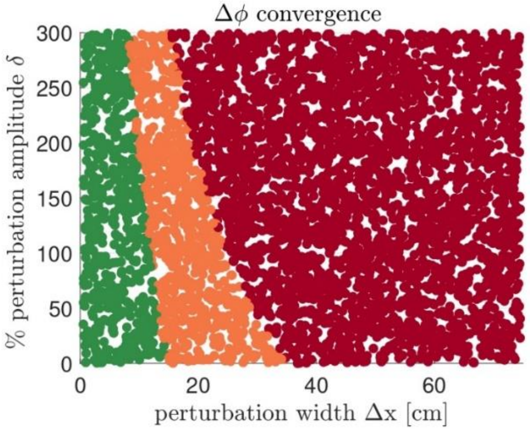
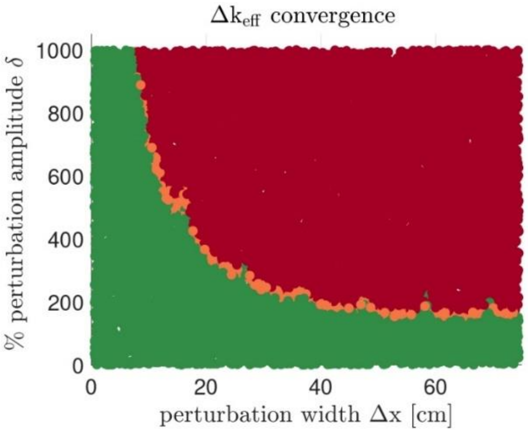
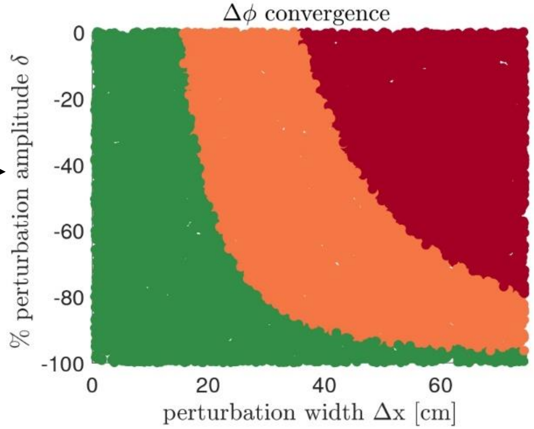
$\nu\Sigma_{f,1}$

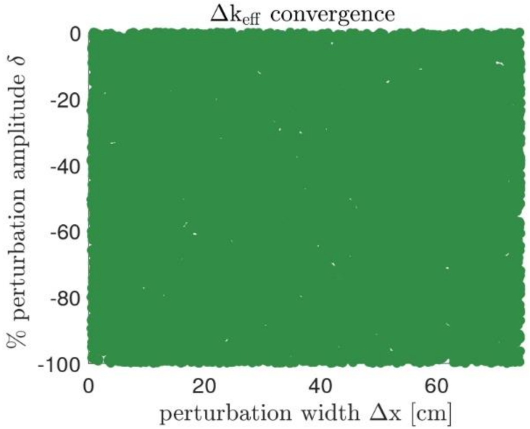
Thermal fission perturbation











Main results

- Both analytic and numerical cases highlighted GPT convergence limits
- We analysed simple cases, yet they were adequate to prove mathematically that GPT may not work for all kind of perturbations
- The product $\delta\Delta x$ seems to roughly delimit the convergence region for cross section data perturbation
- Perturbation position x_0 seems to have a negligible influence on the convergence region

Further developments

- **The effects related to the perturbation of the emission spectrum χ (fast)** on convergence are neglected in this two-group model
- The effect of **degenerate eigenvectors** can be seen in 2D/3D geometry only
- The effect of **general, superimposed perturbations** $\delta\hat{L}$ and $\delta\hat{F}$ has to be verified
- The determination of a quantitative way to assess whether a perturbation is sufficiently small to be handled by GPT is a major issue

Thank you for your attention!

Any questions?

Backup

GPT recursive formulas

- Weighting coefficients for the expansion,

$$a_m^{(n)} = \frac{1}{(\mu_m - \mu_0)\left\langle \vec{\varphi}_m^+ \middle| \hat{F}\vec{\varphi}_m \right\rangle} \left[\sum_{k=1}^{n-1} \left[\lambda^{(k)} a_m^{(n-k)} \left\langle \vec{\varphi}_m^+ \middle| \hat{F}\vec{\varphi}_m \right\rangle + \sum_{i=0}^{\infty} \lambda^{(k)} a_i^{(n-k-1)} \left\langle \vec{\varphi}_m^+ \middle| \delta\hat{F}\vec{\varphi}_i \right\rangle \right] + \right.$$
$$\left. - \sum_{i=0}^{\infty} a_i^{(n-1)} \left\langle \vec{\varphi}_m^+ \middle| (\delta\hat{L} - \lambda^{(0)}\delta\hat{F})\vec{\varphi}_i \right\rangle \right]$$

GPT recursive formulas

- Eigenvalue perturbations,

$$\lambda^{(n)} = \frac{\left\langle \vec{\varphi}_0^{+} \middle| (\delta\hat{L} - \lambda^{(0)}\delta\hat{F})\vec{\phi}^{(n-1)} \right\rangle - \sum_{k=1}^{n-1} \left\langle \vec{\varphi}_0^{+} \middle| \lambda^{(k)}\hat{F}\vec{\phi}^{(n-k)} \right\rangle - \sum_{k=1}^{n-1} \left\langle \vec{\varphi}_0^{+} \middle| \lambda^{(k)}\delta\hat{F}\vec{\phi}^{(n-k-1)} \right\rangle}{\left\langle \vec{\varphi}_0^{+} \middle| \hat{F}\vec{\varphi}_0 \right\rangle}$$